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**Vector Space**

* A set of vectors {Ψ1, Ψ2, Ψ3,…}and associate scalars {a­1, a­2, a3,…}
* Has vector addition and scalar multiplication

**Vector addition:**

* Commutative
* Associative
* For each Ψi, there is a vector Θ such that Ψi + Θ = Θ + Ψi = Ψi
* Inverse property: For each Ψi, there is a vector (-Ψi) such that:

Ψi + (-Ψi ) = Θ

**Scalar Multiplication:**

* Distributive
* Associative
* Zero and Identity

**Hilbert Space**

1. A Linear Vector Space
2. Has inner products that establish certain conditions

* 1. **Conjugate Symmetry**
  2. **Linearity** with respect to(w/rt) 2nd vector
  3. **Antilinear** w/rt. 1st vector
  4. A.K.A. **Positive Definiteness**

1. Hilbert spaces are **separable,**
2. Complete or “no gaps”

Cauchy Sequence

**e.g.** ℝ is separable because ℚ is a countable, dense subset. ℚ is incomplete. Because of 3.1< π < 3.2. The “gap” is where π is.

**Types of Hilbert Spaces**

1. **Finite-Dimensional Hilbert Space:** : n basis vectors

Inner product on : typical dot product.

Inner product on : complex inner product.

1. **Infinite Dimensional Hilbert Spaces:**

e.g. Vector space of complex-valued functions, w/ inner product:

Above eqn. need square-integrable functions because:

Inner product doesn’t exist\*\*\*

Square-integrable functions:

**1st Postulate of Quantum Mechanics: Physical State of the system is represented by a vector in Hilbert Space.**

e.g. , related to the probability of finding a particle at at the time t. Thus, this represents a state/condition of the system. So, they can represent vectors in Hilbert Space

e.g. Spin state: vectors in (a Hilbert space)

**Dirac Notation:**

Ket: , a vector. E.g.

Bra: , e.g. if , then

If , then

Braket: , represents inner product in Hilbert Space.

e.g. if , then

e.g. if , then

**Properties of bras, kets, and brakets:**

1. Every has a
2. **Constant multiple property:** and
3. Braket properties:
   1. **Conjugate Symmetry**:
   2. **Linearity** w.r.t. **2nd vector**:
   3. A**ntilinearity** **w.r.t. 1st vector**:
   4. Norm is positive definite: , only when

**Other Properties:**

1. **Triangle Inequality**

-Inequality only when linearly dependent.

1. **Schwarz Inequality**
2. **Orthogonality & Orthonormality**

orthonormal=\begin{cases}\langle\Psi|\phi \rangle=0\\ \langle\Psi|\Psi\rangle=1, \langle\phi|\phi\rangle=1\end{cases}

**Operators:**

, = an operator

In function space, operators aren’t generally written as matrices like below:

**Properties of Operators:**

**Linear Operators:**

**a.**

**b.**

**c. Expectation value of operator with respect to a state |Ψ⟩ :**

**d. |Ψ⟩ ⟨Φ| is a linear operator.**

e.g.

**Types of Operators in Quantum Mechanics**

1. **Inverse Operators**

**Properties:**

1. **Hermitian Operators**
2. **Anti-Hermitian Operator**
3. **Unitary Operators**
4. **Projection Operator: must satisfy two conditions**

e.g.  **is a projection operator**

**properties:**

1. **If and , and then, and are projection operators.**
2. **If and are orthogonal projection operators, then**
3. **Sum of projection operators:**

**CAN be a projection operator IF AND ONLY IF all in this sum are MUTUALLY ORTHOGONAL**

**Hermitian Adjoint/Conjugate**

**For a scalar a, (simple complex conjugate)**

, for any Ψ and Φ

**Another definition for :**

|  |  |
| --- | --- |
| **1.**  **2.**  **3.**  **4.** | **5.**  **6.**  **7.**  **8.** |

**Above are the properties**

**Properties:  
1.**

**2.**

**3.**

**4.**

**5.**

**6.**

**7.**

**8.**

E.g.

Check:

**Problem 1.17** A particle is represented (at a time t = 0) by the wave function

\Psi(x,0) = \begin{cases} A(a^{2}–x^{2}) & \mbox{if } -a\leq x \leq +a \\ 0, & \mbox{otherwise}\end{cases}

1. Determine the normalization constant A.
2. What is the expectation value of x (at time t = 0)?
3. What is the expectation value of p (at time t = 0)? (Note you cannot get it from p = md<x>/dt. Why not?)
4. Find the expectation value of
5. Find the expectation value of
6. Find the uncertainty in .

**Answers to Problem 1.17:**

1. Normalization Constant Condition:

\displaystyle\int\_{-\infty}^{\infty} |\Psi|^{2} dx = 1

Plug in

\displaystyle\int\_{-a}^{\a}[A(a^{2}-x^{2})]^{2} dx = 1

A^{2}\displaystyle\int\_{-a}^{\a}(a^{4}-2a^{2}x^{2}+x^{4}) dx = 1

\left. A^{2}\left[\frac{2}{4}\right] \right|\_{-a}^{a} = 1

1. Find

is the scalar x and everywhere except on [-a,a]

1. Find

But

1. Find
2. Find
3. Find
4. Find
5. Check if consistent

\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}

\*\*\*\*\* \*\*\*\*\*

Just remove the ‘b’ in ‘bmatrix’ so it would work…

\*\*\*\*\* \*\*\*\*\*

\left[ \begin{array}{c} x\_1 \\ x\_2 \end{array} \right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \left[ \begin{array}{c} y\_1 \\ y\_2 \end{array} \right]

Ñ

2.6.1

Ques­tion:

A ma­trix $A$ is de­fined to con­vert any vec­tor ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $x{\hat\imath}+y{\hat\jmath}$ into ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $2x{\hat\imath}+4y{\hat\jmath}$. Ver­ify that ${\hat\imath}$ and ${\hat\jmath}$ are or­tho­nor­mal eigen­vec­tors of this ma­trix, with eigen­val­ues 2, re­spec­tively 4.

An­swer:

Take $x$ $\vphantom0\raisebox{1.5pt}{$=$}$ 1, $y$ $\vphantom0\raisebox{1.5pt}{$=$}$ 0 to get that ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ ${\hat\imath}$ trans­forms into ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $2{\hat\imath}$. There­fore ${\hat\imath}$ is an eigen­vec­tor, and the eigen­value is 2. The same way, take $x$ $\vphantom0\raisebox{1.5pt}{$=$}$ 0, $y$ $\vphantom0\raisebox{1.5pt}{$=$}$ 1 to get that ${\hat\jmath}$ trans­forms into $4{\hat\jmath}$, so ${\hat\jmath}$ is an eigen­vec­tor with eigen­value 4. The vec­tors ${\hat\imath}$ and ${\hat\jmath}$ are also or­thog­o­nal and of length 1, so they are or­tho­nor­mal.

In lin­ear al­ge­bra, you would write the re­la­tion­ship ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $A{\skew0\vec r}$ out as:

In short, vec­tors are rep­re­sented by columns of num­bers and ma­tri­ces by square ta­bles of num­bers.

2.6.2

Ques­tion:

A ma­trix $A$ is de­fined to con­vert any vec­tor ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(x,y)$ into the vec­tor ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(x+y,x+y)$. Ver­ify that $(\cos 45^\circ ,\sin 45^\circ)$ and $(\cos 45^\circ ,-\sin 45^\circ)$ are or­tho­nor­mal eigen­vec­tors of this ma­trix, with eigen­val­ues 2 re­spec­tively 0. Note: $\cos 45^\circ$ $\vphantom0\raisebox{1.5pt}{$=$}$ $\sin 45^\circ$ $\vphantom0\raisebox{1.5pt}{$=$}$ $\frac 12\sqrt{2}$.

An­swer:

For ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\cos 45^\circ ,\sin 45^\circ)$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\frac 12\sqrt{2},\frac 12\sqrt{2})$, $x$ $\vphantom0\raisebox{1.5pt}{$=$}$ $y$ $\vphantom0\raisebox{1.5pt}{$=$}$ $\frac 12\sqrt{2}$ so ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\sqrt{2},\sqrt{2})$, and that is twice ${\skew0\vec r}$. For ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\cos 45^\circ ,-\sin 45^\circ)$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\frac 12\sqrt{2},-\frac 12\sqrt{2})$, $x$ $\vphantom0\raisebox{1.5pt}{$=$}$ $-y$ $\vphantom0\raisebox{1.5pt}{$=$}$ $\frac 12\sqrt{2}$ so ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ (0,0), and that is zero times ${\skew0\vec r}$.

The square length of ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\cos 45^\circ ,\sin 45^\circ)$ is ${\skew0\vec r}\cdot{\skew0\vec r}$, which is given by the sum of the square com­po­nents: $\cos^245^\circ +\sin^245^\circ$. That is one, so the vec­tor is of length one. The same for ${\skew0\vec r}$ $\vphantom0\raisebox{1.5pt}{$=$}$ $(\cos 45^\circ ,-\sin 45^\circ)$. The dot prod­uct of $(\cos 45^\circ ,\sin 45^\circ)$ and $(\cos 45^\circ ,-\sin 45^\circ)$ is $\cos^245^\circ -\sin^245^\circ$. That is zero, be­cause $\cos 45^\circ$ $\vphantom0\raisebox{1.5pt}{$=$}$ $\sin 45^\circ$, so the two eigen­vec­tors are or­thog­o­nal.

In lin­ear al­ge­bra, you would write the re­la­tion­ship ${\skew0\vec r}_2$ $\vphantom0\raisebox{1.5pt}{$=$}$ $A{\skew0\vec r}$ out as: